TRINITY COLLEGE

IN NOMINE DOMINI

Trial WACE Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C/3D
Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________________

Your name Solutions

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	6	6	50	40	$33\frac{1}{3}$
Section Two: Calculator-assumed	13	13	100	80	$66\frac{2}{3}$
			Total	120	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
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3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section One: Calculator-free **(40 Marks)**

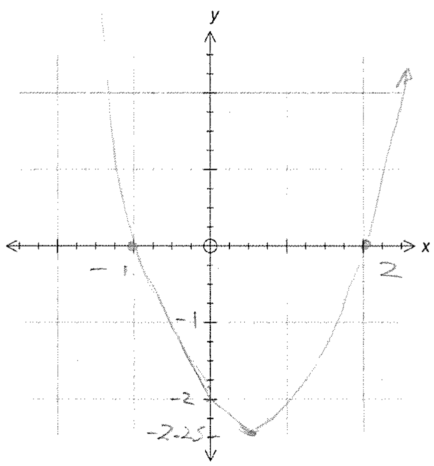
This section has **six (6)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

(a) Sketch a graph of the function $y=(x-2)(x+1)$ on the axes below.

(1 mark)



(b) Determine the area trapped between $y=(x-2)(x+1)$, the x-axis, the y-axis and the line $x=-3$.

(4 marks)

$y=(x-2)(x+1)=x^2-x-2$

$$=\int_{-3}^{-1} x^2-x-2\,dx-\int_{-1}^{0} x^2-x-2\,dx$$

$$=\left.\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{-1}-\left.\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{0}$$

$$=\left[\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(-9-\frac{9}{2}+6\right)\right]-\left[(0)-\left(-\frac{1}{3}-\frac{1}{2}+2\right)\right]$$

$$=\left(\frac{7}{6}+\frac{15}{2}\right)+\frac{7}{6}$$

$$=\frac{59}{6}=9\frac{5}{6}$$

Question 2 **(9 marks)**

(a) Determine the equation of the tangent to the curve $x^3 - 4xy + y^3 = 1$ at the point (1, -2). (5 marks)

$3x^2 - 4y - 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$ ✓

$\frac{dy}{dx}(3y^2 - 4x) = 4y - 3x^2$

$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$ ✓

at (1, -2) $\frac{dy}{dx} = \frac{-8-3}{12-4} = -\frac{11}{8}$

✓

eqn tangent

$11x + 8y = c$ ✓

$11 - 16 = c$

$-5 = c$

$11x + 8y = -5$ ✓

(b) Evaluate $\int_{\sqrt{e}}^{e} \frac{1}{x(\ln x)^2}dx$ using the substitution $u = \ln x$. (4 marks)

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x.du = dx$ ✓

when $x = \sqrt{e}$

$u = \frac{1}{2}$ ✓

when $x = e$

$u = 1$

$\int_{0.5}^{1} \frac{1}{\cancel{x}(u^2)} . \cancel{x}\, du$ ✓

$= \int_{0.5}^{1} u^{-2} du$

$= -u^{-1}\Big]_{0.5}^{1}$

$= -\frac{1}{u}\Big]_{\frac{1}{2}}^{1}$ ✓

$= -1 - (-2)$

$= 1$

Question 3 **(5 marks)**

Evaluate $\int_0^{\pi} (9\sin x - 12\sin^3 x)dx$

$= \int_0^{\pi} (9\sin x - 12\sin x \sin^2 x)dx$

$= \int_0^{\pi} 9\sin x - 12\sin x(1-\cos^2 x)\ dx$

$= \int_0^{\pi} -3\sin x + 12\sin x\cos^2 x\quad dx$

$y = \cos^3 x$

$\frac{dy}{dx} = -3\cos^2 x \sin x$

$= \left. 3\cos x - 4\cos^3 x \right]_0^{\pi}$

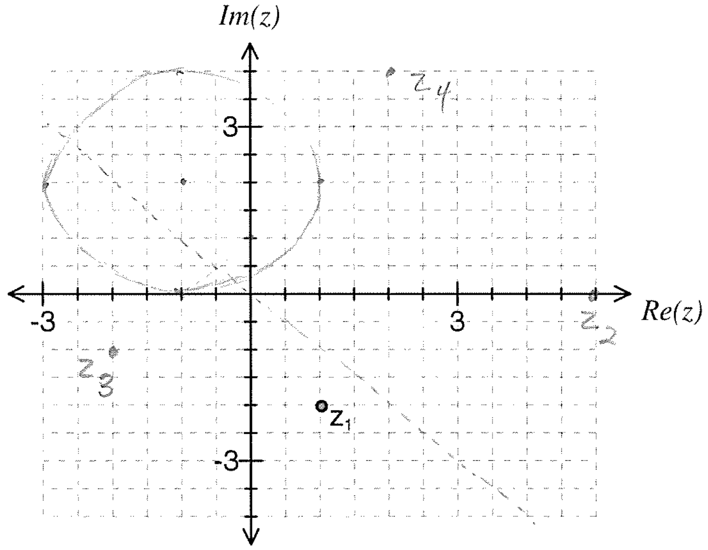
$= [-3+4] - [3-4]$

$= 1 + 1$

$= 2$

Question 4 **(7 marks)**

The Argand diagram below shows the complex number z_1.



(a) On the same diagram plot and label the complex numbers given by (3 marks)

$z_2 = z_1\overline{z_1}$ $(1-2i)(1+2i) = 1+4 = 5$

$z_3 = i^3 z_1$ $-i(1-2i) = -2-i$

$z_4 = 10z_1^{-1}$ $10\,\frac{1}{1-2i} \times \frac{1+2i}{1+2i} = \frac{10(1+2i)}{5} = 2+4i$

(b) On the same diagram sketch the region given by $|z + z_1| \le 2$. (2 marks)

$|z-(-1+2i)| \le 2$

(c) The locus of the complex number $z = x+iy$ satisfies the inequation $|z+1| < |z-i|$.
Determine the equation of the locus in the form $y > f(x)$ or $y < f(x)$. (2 marks)

$y < -x$

Question 5 **(5 marks)**

Use the identity $2i\ \sin n\theta = z^n - \dfrac{1}{z^n}$ to prove that $\sin^3\theta = \dfrac{3\ \sin\ \theta - \sin\ 3\theta}{4}$.

$$(2i\sin\theta)^3 = \left(z - \frac{1}{z}\right)^3$$

$$-8i\sin^3\theta = z^3 - 3z^2\frac{1}{z} + 3z\frac{1}{z^2} - \frac{1}{z^3}$$

$$= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$$

$$-8i\sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$$

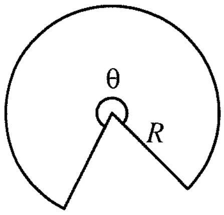
$$\sin^3\theta = \frac{2i\ (\sin 3\theta - 3\sin\theta)}{-8i}$$

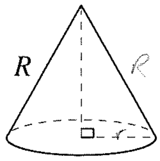
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

QED

Question 6 **(9 marks)**

A minor sector of angle $2\pi - \theta$ is removed from a circular piece of paper of radius R. The two straight edges of the remaining major sector are pulled together to form a right circular cone, with a slant height of R.





(a) Show that the volume of the cone is given by $V = \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$. (4 marks)

$2\pi r = R\theta$

$r = \dfrac{R\theta}{2\pi}$

$h = \sqrt{R^2 - r^2}$

$= \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2}$

$V = \frac{1}{3}\pi r^2 . h$

$V = \frac{1}{3}\pi\left(\frac{R\theta}{2\pi}\right)^2 . \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2}$

$= \dfrac{\pi}{3}\dfrac{R^2\theta^2}{4\pi^2} . \sqrt{\dfrac{4\pi^2R^2 - R^2\theta^2}{4\pi^2}}$

$= \dfrac{\cancel{\pi}}{3}\dfrac{R^2\theta^2}{4\pi^2} . \dfrac{\sqrt{4\pi^2R^2 - R^2\theta^2}}{2\cancel{\pi}}$

$= \dfrac{R^2\theta^2 . R\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$

$= \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$

(b) Determine the value of θ which maximises the volume of cone. (5 marks)

$$V = \frac{R^3\theta^2}{24\pi^2} \cdot \sqrt{4\pi^2 - \theta^2}$$

$$\frac{dV}{d\theta} = \frac{R^3\theta^2}{24\pi^2} \cdot \frac{1}{2}(4\pi^2-\theta^2)^{-\frac{1}{2}} \cdot (-2\theta) + (4\pi^2-\theta^2)^{\frac{1}{2}} \cdot \frac{2R^3\theta}{24\pi^2}$$

Max when $\frac{dV}{d\theta} = 0$

$$0 = -\frac{R^3\theta^3}{24\pi^2(4\pi^2-\theta^2)^{\frac{1}{2}}} + \frac{2R^3\theta}{24\pi^2} \cdot (4\pi^2-\theta^2)^{\frac{1}{2}}$$

$$0 = -R^3\theta^3 + 2R^3\theta(4\pi^2-\theta^2)$$

provided

$4\pi^2 - \theta^2 \neq 0$

$4\pi^2 \neq \theta^2$

$2\pi \neq \theta$

$$R^3\theta^3 = 2R^3\theta(4\pi^2-\theta^2)$$

$$\theta^2 = 8\pi^2 - 2\theta^2$$

$$3\theta^2 = 8\pi^2$$

$$\theta = \frac{2\sqrt{2}\pi}{3}$$

End of questions

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MATHEMATICS SPECIALIST 3C/3D

Section Two: Calculator-assumed

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Student Number: In figures | | | | | | | | |

In words ______________________

Your name Solutions

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

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Section Two: Calculator-assumed **(80 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 7 **(6 marks)**

The temperature, I °C, of a liquid in an insulated flask at any time t seconds can be described by the differential equation $\frac{dI}{dt} = -0.003I$.

(a) How long will it take for the liquid in the flask to fall by 10%? (2 marks)

$A = A_0 e^{-0.003t}$

$0.9 = e^{-0.003t}$ ✓

$t = 35.12$ ✓

It would take just over 35 sec.

The temperature of a liquid in another, uninsulated, flask is falling exponentially at a percentage decay rate of 0.75%.

(b) If the initial temperatures of the liquids in the insulated and uninsulated flasks are 65°C and 95°C respectively, determine when the difference in temperature between the two liquids is 10°C. (4 marks)

Insulated

$A = 65e^{-0.003t}$

uninsulated

$A = 95e^{-0.0075t}$ ✓

$65e^{-0.003t} - 95e^{-0.0075t} = 10$ at $t = 47.99$ ✓

$95e^{-0.0075t} - 65e^{-0.003t} = 10$ at $t = 144.55$ and $t = 587.34$ ✓✓

Difference is 10°C when $t \approx 48, 145, 587$ sec

Question 8 **(5 marks)**

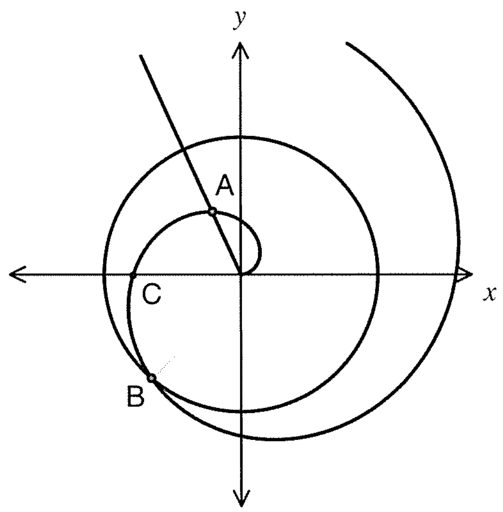
(a) Calculate the distance between the points with polar coordinates $\left(5, \frac{2\pi}{3}\right)$ and $\left(12, -\frac{5\pi}{6}\right)$, where distances are in centimetres and angles are in radians. (1 mark)

$\frac{2\pi}{3} - \frac{5\pi}{6} = \frac{9\pi}{6} = \frac{3\pi}{2}$

ie angle $= 90°$

$\therefore$ distance $= \sqrt{5^2+12^2}$
$= 13$ cm

(b) The graphs of $\theta = \alpha$, $r = b$ and $r = n\theta$ are shown below together with the points A and B which have polar coordinates of $(1, 2)$ and $(b, 4)$.



r, θ
A $(1, 2)$
B $(b, 4)$

(i) Determine the values of α, b, n.

(3 marks)

$(1, 2)$ $\theta = 2$

$r = n\theta$
$1 = n \times 2$
$\frac{1}{2} = n$ ✓
$\alpha = 2$ ✓

$(b, 4)$ $b = 0.5 \times 4 = 2$
$b = 2$ ✓

(ii) Calculate the polar coordinates of point C.

(1 mark)

$r = 0.5\,\theta$
$r = \frac{1}{2}\pi = \frac{\pi}{2}$

$C\left(\frac{\pi}{2}, \pi\right)$

Question 9 **(8 marks)**

The point A has position vector $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Determine the value of a if the vectors $\overrightarrow{OA}$ and $a\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ are perpendicular. (1 mark)

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ 3 \\ -3 \end{bmatrix} = 0$$

$3a = 18$, $a = 6$ ✓

(b) Calcuate the size of the angle between $\overrightarrow{OA}$ and the z-axis, to the nearest degree.(2 marks)

use calculator angle([3, -2, 4], [0, 0, 1]) = 42.03 ✓✓

(c) Calculate the value of b if the point (7, b, 2) lies in the plane containing the point (-1, 2, 5) and with normal vector $\overrightarrow{OA}$. (2 marks)

$a.n = r.n$

$$\begin{bmatrix} 7 \\ b \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$b = 8$

(d) Determine the value of c if the point (15, -14, c) lies on the straight line through A and the point (-1, 2, 5). (3 marks)

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -14 \\ c \end{bmatrix} \checkmark$$

$3 + 4\lambda = 15$

$\lambda = 3$

$4 - 3 = c$

$\underline{1 = c}$ ✓

Question 10 **(6 marks)**

The transformation matrix $M = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$.

M represents a shear of factor k parallel to the y-axis followed by a rotation of 90° clockwise.

(a) Use the geometric properties of the two transformations to explain why $|M| = 1$ (1 mark)

shear same area, rotation same area

$\therefore |M| = 1$

(b) Determine the values of a, b and k. (3 marks)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$$

$k = -2, \quad a = -1 \quad b = 0$

(c) The point P is transformed by M to the point (8, 3). Determine the coordinates of P. (2 marks)

$$\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$-2a + b = 8$

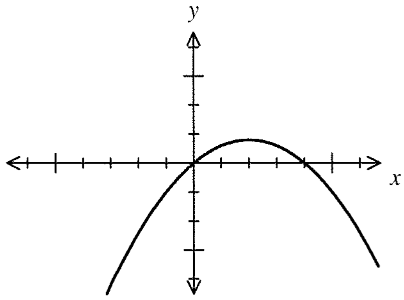
$-a = 3$

$a = -3, \quad b = 2$

$P(-3, 2)$

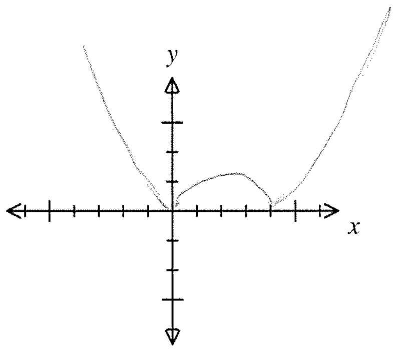
Question 11 **(5 marks)**

The graph of $y = f(x)$ is shown below.

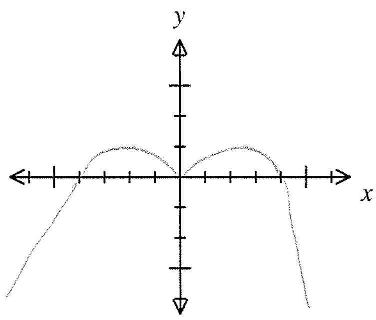


(a) Sketch the graphs of: (1mark)

(i) $y = |f(x)|$



(ii) $y = f(|x|)$ (1mark)



(b) The equation $|ax + b| = |x - 4|$ has solutions $x = -0.2$ and $x = -3$. Calculate the values of a and b. (3 marks)

$|-0.2a + b| = |-4.2|$

$|-3a + b| = |-7|$

$(-4, -5)$, $(-1, 4)$, $(1, -4)$, $(4, 5)$

Question 12 **(4 marks)**

Prove by deduction that $\dfrac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta}=\tan\theta$.

$$= \frac{1+2\sin\theta\cos\theta-(1-2\sin^2\theta)}{1+2\sin\theta\cos\theta+(2\cos^2\theta-1)}$$

$$= \frac{1+2\sin\theta\cos\theta-1+2\sin^2\theta}{1+2\sin\theta\cos\theta+2\cos^2\theta-1}$$

$$= \frac{2\sin\theta(\sin\theta+\cos\theta)}{2\cos\theta(\sin\theta+\cos\theta)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

QED

Question 13 **(6 marks)**

Two complex numbers are given by $u = 3i$ and $v = \dfrac{3\sqrt{3} - 3i}{2}$.

(a) Express $u^3 v$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi \le \theta \le \pi$ and $r \ge 0$. (2 marks)

$81(\text{cis } 240°)$

$= 81\left(\cos\frac{-2\pi}{3} + \sin\frac{-2\pi}{3} i\right)$

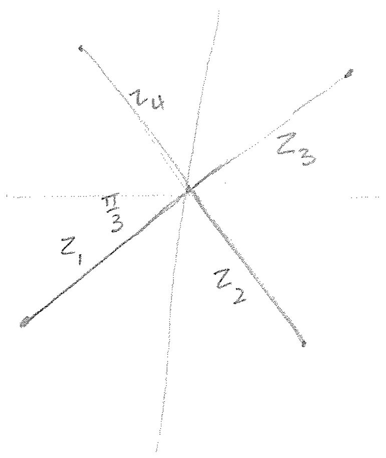
(b) Determine all solutions for z in the form $re^{i\theta}$, given that $z^4 = u^3 v$. (2 marks)

(each angle $\frac{\pi}{2}$ apart)

$z_1 = 3e^{-\frac{2\pi}{3}i}$, $z_2 = 3e^{-\frac{\pi}{6}i}$, $z_3 = 3e^{\frac{\pi}{3}i}$, $z_4 = 3e^{\frac{5\pi}{6}i}$

✓ ✓

(c) Show that the sum of all the solutions from part (b) is 0. (2 marks)



z_1 and z_3 opposite direction, same magnitude

z_2 and z_4 opposite direction, same magnitude

Question 14 **(5 marks)**

At a school with 108 boarders, boarders can either eat breakfast or not. The canteen manager estimates that of those boarders who eat breakfast one morning, 5% of them will not eat breakfast the next morning and of those boarders who do not eat breakfast one morning, 55% of them eat breakfast the following morning.

(a) If 55 boarders eat breakfast on Monday, how many boarders should the canteen manager expect to eat breakfast on Wednesday? (3 marks)

$$T = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \qquad P = \begin{bmatrix} 55 \\ 53 \end{bmatrix}$$

$$T^2 P = \begin{bmatrix} 91.96 \\ 16.04 \end{bmatrix}$$

92 Boarders for Breakfast

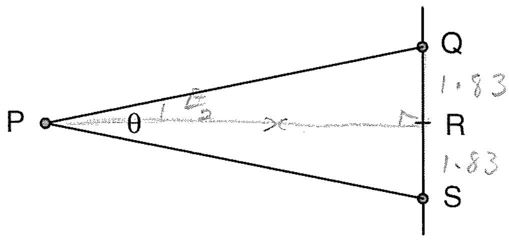
(b) In the long term, what proportion of boarders can be expected to eat breakfast? (2 marks)

$$\frac{0.55}{0.55 + 0.05} = \frac{11}{12}$$

91.67%

Question 15 **(6 marks)**

The diagram shows a hockey player at P running directly towards R, the midpoint of QS, where Q and S are the goalposts spaced 3.66m apart at one end of a hockey field. PR is perpendicular to QS and θ, the shooting angle, is the size of angle QPS.



If the player is running at a constant speed of 6m/s towards R, at what rate is the shooting angle θ increasing at the instant when the player is 9m from R?

Give your answer in degrees per second rounded to 2 decimal places.

$\frac{dx}{dt} = -6\text{m/s}$ ✓

$\tan\left(\frac{\theta}{2}\right) = \frac{1.83}{x}$ ✓

let $\frac{\theta}{2} = \alpha$.

$\frac{d\alpha}{dt} = \frac{d\alpha}{dx} \cdot \frac{dx}{dt}$

$\tan\alpha = \frac{1.83}{x}$

$= \frac{-1.83}{9^2} \cos^2(11.5°) \times (-6)$ ✓

$\frac{1}{\cos^2\alpha} \cdot \frac{d\alpha}{dx} = -\frac{1.83}{x^2}$ ✓

$\frac{d\alpha}{dt} = 0.13\ ^R/\text{sec}$

$\frac{d\alpha}{dx} = \frac{-(1.83)\cos^2\alpha}{x^2}$

$\tan\alpha = \frac{1.83}{9}$

$\angle\alpha = 11.49°$

$\frac{d\theta}{dt} = 0.26\ ^R/s$ ✓

$= 14.92°/\text{sec}$ ✓

Question 16 **(5 marks)**

(a) The matrix equation $AX = B$ could be used to solve the following system of equations.

$$2a + 3b = c - 5$$
$$b - 2c - 4a = 1$$
$$5 = c + b$$

Write down suitable matrices for A, X and B. (DO NOT SOLVE YOUR EQUATIONS) (2 marks)

$A = \begin{bmatrix} 2 & 3 & -1 \\ -4 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$B = \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$

(b) If $PQ = 3P + I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 7 & -9 \\ 5 & -8 \end{bmatrix}$, determine matrix P. (3 marks)

$PQ = 3P + I$

$PQ - 3P = I$

$P(Q - 3I) = I$

$P = I(Q - 3I)^{-1}$

$P = \begin{bmatrix} -11 & 9 \\ -5 & 4 \end{bmatrix}$

Question 17 **(7 marks)**

The displacement $x(t)$ metres, of a small particle undergoing simple harmonic motion is given by $x(t) = A\cos\omega t + B\sin\omega t$, where A, B and ω are positive constants.

(a) Show that $x''(t) + \omega^2 x(t) = 0$. (2 marks)

$x'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$

$x''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$

$= -\omega^2 (A\cos\omega t - B\sin\omega t)$

$= -\omega^2 x(t)$

$x''(t) + \omega^2 x(t) = 0$

The body passes through the origin (where $x(t) = 0$) five times per second, $x(0) = 1.5$ m and $x'(0) = 7.5$ ms^{-1}.

(b) Determine the exact values of the constants A, B and ω. (3 marks)

$\omega = 2\pi \times 2.5 = 5\pi$

$1.5 = A\cos(0) + B\sin 0$

$1.5 = A$

$7.5 = -5\pi(1.5\sin 0 - B\cos 0)$

$B = \frac{3}{2\pi}$

(c) What is the amplitude of motion, correct to the nearest millimetre? (2 marks)

amp $= \sqrt{1.5^2 + \left(\frac{3}{5\pi}\right)^2}$

$= 1.574$ m

Question 18 **(8 marks)**

Every odd integer I can be written as $I = 10n + c$, where n is an integer and $c = 1, 3, 5, 7, 9$.

(a) Show how the integers 237, 3 and -35 can be written this way. (3 marks)

$237 = 10 \times 23 + 7$

$3 = 10 \times 0 + 3$

$-35 = 10 \times (-4) + 5$

(b) By considering the five different cases for c, or otherwise, prove that the square of every odd integer ends in 1, 5 or 9. (5 marks)

$c = 1$

$(10n+1)^2 = 100n^2 + 20n + 1 = 10(10n^2+2n) + 1$

$c = 3$

$(10n+3)^2 = 100n^2 + 60n + 9 = 10(10n^2+6n) + 9$

$c=5$

$(10n+5)^2 = 100n^2 + 100n + 25 = 10(10n^2+10n+2) + 5$

$c = 7$

$(10n+7)^2 = 100n^2 + 140n + 49 = 10(10n^2+14n+4) + 9$

$c = 9$

$(10n+9)^2 = 100n^2 + 180n + 81 = 10(10n^2+18n+8) + 1$

all cases end with 1, 5, 9

Question 19 **(9 marks)**

Relative to itself, an anti-ballistic missile (ABM) launch site detects a ballistic missile at $11\mathbf{i} - 18\mathbf{j} + 10\mathbf{k}$ km headed at constant velocity for a target at $35\mathbf{i} + 14\mathbf{j} + \mathbf{k}$ km. The ballistic missile is expected to hit the target in 50 seconds.

(a) How close does the ballistic missile come to the ABM launch site? (5 marks)

$$AB = \begin{bmatrix} 35-11 \\ 14+18 \\ 1-10 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} \qquad V = \frac{1}{50}\begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.64 \\ -0.18 \end{bmatrix} \text{km/s}$$

closest when $(OA + t.V_m) \cdot V_m = 0$

$$\begin{bmatrix} 11+0.48t \\ -18+0.64t \\ 10-0.18t \end{bmatrix} \cdot \begin{bmatrix} 0.48 \\ 0.64 \\ -0.18 \end{bmatrix} = 0 \qquad t = 11.96\text{ sec}$$

at $t = 11.96$

distance $= 21.19$ km

(b) The launch site plans to fire an ABM to hit the ballistic missile. The hit is timed to take place at the instant the ballistic missile comes within 8km of the target. Assuming the ABM instantly achieves a constant velocity of 1150 ms^{-1} as it is launched, how long from the time of detection should the defence site fire it? (4 marks)

$AB = 41$ km

1150 m/s $= 1.15$ km/s

8km from A = 33km from B

$\frac{33}{41} \times 50 = 40.2$ sec

$$OC = \begin{bmatrix} 11 \\ -18 \\ 10 \end{bmatrix} + \frac{33}{41}\begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} = \begin{bmatrix} 30.32 \\ 7.76 \\ 2.76 \end{bmatrix}$$

$|OC| = 31.42$ km

Time $= 31.42 \div 1.15 = 27.32$ sec

must launch $40.2 - 27.32 = 12.88$ sec

End of questions